Indian Statistical Institute
Bangalore Centre
B.Math (Hons.) III Year 2015-2016
First Semester
Statistics III

Back paper Examination Date : 31.12.15

Answer as many questions as possible. The maximum you can score is 100.
All symbols have their usual meaning, unless stated otherwise.
State clearly the results you use.

1. Consider a random $n \times 1$ vector X with $E(X) = \mu$ and $COV(X) = \Sigma$. For an $m \times n$ matrix A, show that $E(X'AX) = \mu' A\mu + tr(A\Sigma)$. [5]

2. Define multivariate normal distribution. Consider normal variates $x_0, x_1, \cdots x_n$ with mean $\mu_0 = 0, \mu_1, \cdots \mu_n$ and same variance σ^2. Suppose $x_1, \cdots x_n$ are independent of each other, while the correlation between x_0 and x_i is ρ for each i.

   Define $y_i = x_i - ax_0$. Show that $Y = (y_1, \cdots y_n)'$ follows multivariate normal distribution. Find its mean and covariance matrix. [1 + 5 = 6]

3. Consider a random vector $X = (X_1, \cdots X_p)'$.

   (a) Find the 'best predictor' of X_1 among **all linear** functions of $X_2, \cdots X_p$.

   (b) Denote 'the best linear predictor' of X_1 obtained in (a) by $X_{1.2\cdots p}$. Denote the correlation between X_1 and $X_{1.2\cdots p}$ by $\rho_{1.2\cdots p}$. Show that

   $$1-\rho^2_{1.2\cdots p} = det\rho_{11}/det\rho_{22}, \ where \ \rho_{tt} = ((corr(X_i, X_j)))_{t\leq i,j\leq p}, \ t = 1, 2.$$

   [5 + 7 =12]

4. Consider the linear regression model $y_i = \alpha+\beta x_i+\epsilon_i, \ 1 \leq i \leq n$. Here $\epsilon_i, i = 1, 2, \cdots n$ are i.i.d random variables with mean 0 and variance σ^2. Suppose

   $$S_x^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2,$$
   $$S_y^2 = \sum_{i=1}^{n}(y_i - \bar{y})^2 \text{ and}$$
   $$S_{xy} = \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}).$$

   (a) Find least square estimates of the parameters α and β in terms of $\bar{x}, \bar{y}, S_x^2, S_y^2$ and S_{xy}.

   (b) Show that $E(S_{xy}^2) = \sigma^2 S_x^2 + \beta^2 S_x^4$.

   (c) Obtain an unbiased estimator of σ^2. [5 + 5 + 7 =17]

5. (a) Suppose X follows noncentral χ^2 distribution with degrees of freedom k and noncentrality parameter (n.c.p.) λ. Express the p.d.f. of X in terms of p.d.f.s of central χ^2 distributions.

(b) Suppose X and Y are independent and both follow χ^2 distribution with degrees of freedom k and l respectively. If X has noncentrality parameter (n.c.p.) $\lambda \neq 0$ and Y has n.c.p. 0, then show that $X + Y$ follows noncentral χ^2 distribution with degrees of freedom $k + l$ and n.c.p. λ.

(c) Suppose $X_i \sim N(\mu_i, 1)$, $i = 1, 2, \cdots n$ and $X_i's$ are independent. Show that $\sum_{i=1}^{n} X_i^2 \sim \chi^2(n, \lambda)$, $\lambda = \sum_{i=1}^{n} \mu_i^2$.

[2 + 5 + 8 = 15]

6. An experimenter is studying the effect of **different formulations** $A, B, \cdots L$ of an explosive mixture used in the manufacture of dynamite on the **explosive force** . It was felt that the **raw materials from different batches (B)** may be different, leading to difference in the explosive force.

(a) Write down an appropriate linear model in the form given below, explaining every term.

$$Y = \mu 1_n + X_T \tau + X_B \beta + \varepsilon.$$

(b) Consider each of the following linear parametric functions. Either provide an unbiased estimator or prove that it is not estimable.

(i) μ, (ii) τ_1, (iii) $\beta_1 + \beta_2$, (iv) $\tau_1 - \tau_2$.

(c) Prove the following statements.

(i) $1_n \in \mathcal{C}(X_B)$.

(ii) $X_T' X_T$ is a diagonal matrix.

(d) Now suppose the raw materials were taken from b batches chosen at random from a large number of batches.

(i) Write an appropriate model, explaining every term.

(ii) Obtain $Cov(Y)$.

(iii) Define error space and estimation space and obtain them in terms of column spaces of X_T and X_B.

[3 + 8 + (2 + 3) + (3 + 2 + (1 + 2 + 2 + 4)) = 30]

7. Suppose X follows $N_p(0, I_p)$ and A is an $p \times p$ symmetric matrix of rank r.

(a) Prove, without using Fisher-Cochran theorem, that a necessary and sufficient condition for $X'AX$ to follow $\chi^2(r)$ is that A is idempotent.

(b) Suppose $l \in R^P$. Show that if $Al = 0$, then $l'X$ and $X'AX$ are independent.

[8 + 7 = 15]